REMARKS ON IDEAL BOUNDEDNESS, CONVERGENCE AND VARIATION OF SEQUENCES

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Abstract. We answer the questions asked in article [FGT]. The first main result states that for every admissible ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ the quotient space $l^\infty(\mathcal{I})/c_0(\mathcal{I})$ is complete. The second main result states that consistently there is an admissible ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ such that the sets $W(\mathcal{I})$, of all real sequences with finite $\mathcal{I}$-variation, and $c^*(\mathcal{I})$, of all restrictively $\mathcal{I}$-convergent sequences, are equal.

1. Introduction

This paper is a response to remarks and questions from article [FGT]. A proper ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ containing all singletons $\{n\}$, $n \in \mathbb{N}$, will be called *admissible*. We will consider only such ideals. For an ideal $\mathcal{I}$, let $\mathcal{I}^*$ denote its dual filter. The authors of [FGT] consider the vector space $l^\infty(\mathcal{I})$ of all $\mathcal{I}$-bounded sequences $x \in \mathbb{R}^\mathbb{N}$. Recall that $x \in \mathbb{R}^\mathbb{N}$, $x = (x_n)_{n\in\mathbb{N}}$, belongs to $l^\infty(\mathcal{I})$ whenever there exists a set $K \in \mathcal{I}^*$ such that the restriction $x_{\restriction K}$ is bounded in usual sense. A sequence $x \in \mathbb{R}^\mathbb{N}$ is called *$\mathcal{I}$-convergent* to $t \in \mathbb{R}$ if for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} : |x_n - t| \geq \varepsilon\} \in \mathcal{I}$ (if such a t exists then it is unique, and we write $t = \mathcal{I}\text{-}\lim x$). By $c(\mathcal{I})$ we denote the set of all $\mathcal{I}$-convergent sequences, and by $c_0(\mathcal{I})$ – the set of all sequences $\mathcal{I}$-convergent to 0. We say that a sequence $x \in \mathbb{R}^\mathbb{N}$, $x = (x_n)_{n\in\mathbb{N}}$, has *finite $\mathcal{I}$-variation* if there is a set $K = \{k_1 < k_2 < ...\} \in \mathcal{I}^*$ such that

$$\operatorname{Var} x_{\restriction K} := \sum_{n=1}^{\infty} |x_{k_n} - x_{k_{n+1}}| < \infty.$$

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By $W(\mathcal{I})$ we will denote the set of all sequences with finite $\mathcal{I}$-variation. Let $c^*(\mathcal{I})$ stand for the set of all sequences $x \in \mathbb{R}^{\mathbb{N}}$ such that there is a set $K = \{k_1 < k_2 < ...\} \in \mathcal{I}^*$ with $\lim_{n\to\infty} x_{k_n} = l$ for some $l \in \mathbb{R}$. We then say that the sequence x is *restrictively $\mathcal{I}$-convergent* to l. Additionaly put $M(\mathcal{I}) = \{x \in l^\infty(\mathcal{I}) : \exists K \in \mathcal{I}^*(x_{\restriction K} \text{ is monotone})\}$. Note that $M(\mathcal{I}) \subset W(\mathcal{I})$ for any ideal $\mathcal{I}$. The main result of [FGT] states that the following inclusions hold:

$$W(\mathcal{I}) \subset c^*(\mathcal{I}) \subset c(\mathcal{I}) \subset l^\infty(\mathcal{I}).$$

The first part of our paper is devoted to proving the completeness of the quotient space $l^\infty(\mathcal{I})/c_0(\mathcal{I})$. In the second part, we investigate in particular whether there exists an admissible ideal $\mathcal{I}$, for which the equality $W(\mathcal{I}) = c^*(\mathcal{I})$ holds. Using set-theoretic assumption that $\mathfrak{p} = \mathfrak{c}$ we show that there is an admissible $\mathcal{I}$ with $W(\mathcal{I}) = c^*(\mathcal{I})$.

2. Quotient space $l^\infty(\mathcal{I})/c_0(\mathcal{I})$

In [FGT] the following seminorm on $l^\infty(\mathcal{I})$ was introduced:

$$||x||_\infty^{\mathcal{I}} = \inf\{\lambda > 0 : (\exists K \in \mathcal{I}^*)\ (\forall n \in K)\ |x_n| \le \lambda\}.$$

In [FGT, Remark 2] the authors state that for $\mathcal{I}_f$, the ideal of all finite subsets of $\mathbb{N}$, the set $l^\infty(\mathcal{I}_f)$ with the above seminorm is the classical Banach space $l^\infty_{\mathbb{R}}(\mathbb{N})$ of all bounded sequences. This is not exactly true because $||\cdot||_\infty^{\mathcal{I}_f}$ is only a seminorm on $l^\infty(\mathcal{I}_f)$. Note that $||x||_\infty^{\mathcal{I}_f} = 0$ if and only if x is convergent to zero in usual sense. Then, for example, if $x = (1/n)_{n\in\mathbb{N}}$, we have $||x||_\infty^{\mathcal{I}_f} = 0$, while sup-norm of x equals 1. On the other hand, observe that the equality $l^\infty(\mathcal{I}_f) = l^\infty_{\mathbb{R}}(\mathbb{N})$ holds if we consider only the sets without their structures.

The authors of [FGT] ask whether, for any admissible ideal $\mathcal{I}$, the space $l^\infty(\mathcal{I})$ is complete. Since $||\cdot||_\infty^{\mathcal{I}}$ is not a norm, one should reformulate this question in a proper manner. For any admissible ideal $\mathcal{I}$, we define an equivalence relation on $l^\infty(\mathcal{I})$:

$$\forall x, y \in l^\infty(\mathcal{I}) \ \ (x \sim y \Leftrightarrow ||x-y||_\infty^{\mathcal{I}} = 0).$$

Observe that $||x||_\infty^{\mathcal{I}} = 0$ iff x is $\mathcal{I}$-convergent to zero. So, we may consider the quotient normed space $l^\infty(\mathcal{I})/c_0(\mathcal{I})$ consisting of all equivalence classes $[x]_\sim$ for $x \in l^\infty(\mathcal{I})$. Now the question reads as follows: is $l^\infty(\mathcal{I})/c_0(\mathcal{I})$ a Banach space, for any admissible ideal $\mathcal{I}$?

Before answering it, recall some notation. By $\beta\mathbb{N}$ we denote the Čech-Stone compactification of $\mathbb{N}$. For an admissible ideal $\mathcal{I}$, let $P_{\mathcal{I}}$ stand for the set of all proper ultrafilters p in $\mathcal{P}(\mathbb{N})$ such that $\bigcap p = \emptyset$ and $\mathcal{I} \subset p^*$ where p^* means a dual (maximal) ideal to p. In [BS] it is proved that $P_{\mathcal{I}}$ is a closed subset of $\beta\mathbb{N}$. In further considerations we will also use the following fact:

$$\bigcap_{p \in P_{\mathcal{I}}} p = \mathcal{I}^*, \tag{1}$$

whose proof is immediate. Finally, let $C(P_{\mathcal{I}})$ denote the Banach space of all continuous functions $f \colon P_{\mathcal{I}} \to \mathbb{R}$ where a topology in $P_{\mathcal{I}}$ is inherited from $\beta\mathbb{N}$.

Theorem 1. *For every admissible ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$, the spaces $l^\infty(\mathcal{I})/c_0(\mathcal{I})$ and $C(P_{\mathcal{I}})$ are isometrically isomorphic. Consequently, $l^\infty(\mathcal{I})/c_0(\mathcal{I})$ is a Banach space.*

Proof. Let $\mathcal{I}$ be an admissible ideal and $x \in l^\infty(\mathcal{I})$. By [FGT, Proposition 3], for every $p \in P_{\mathcal{I}}$ the notion of p^*-boundedness of x is equivalent to p^*-convergence of x. So, we may define $f_x \colon \mathbb{N} \cup P_{\mathcal{I}} \to \mathbb{R}$ by the formula

$$f_x(p) = \begin{cases} x_p & \text{for } p \in \mathbb{N} \\ p^*\text{-}\lim x & \text{for } p \in P_{\mathcal{I}}. \end{cases}$$

At first, we will show that f_x is continuous. Recall that the topology in $\mathbb{N} \cup P_{\mathcal{I}}$ is generated by a base of the form

$$\{U \cup U^* : U \subset \mathbb{N}\},$$

where $U^* := \{p \in P_{\mathcal{I}} : U \in p\}$. Since every point $p \in \mathbb{N}$ is isolated in $\mathbb{N} \cup P_{\mathcal{I}}$, then f_x is continuous at $p \in \mathbb{N}$. Fix $p \in P_{\mathcal{I}}$. Then $f_x(p) = p^*\text{-}\lim x$, and hence

$$(\forall \varepsilon > 0)\ A(\varepsilon) := \{n \in \mathbb{N} : |x_n - f_x(p)| < \varepsilon\} \in p.$$

We want to show that

$$(\forall \varepsilon > 0)\ (\exists U\text{– a neighborhood of } p)\ \ f_x(U) \subset (f_x(p) - \varepsilon, f_x(p) + \varepsilon).$$

Fix $\varepsilon > 0$. Define $U = A(\frac{\varepsilon}{2}) \cup A(\frac{\varepsilon}{2})^*$. Then the set U is open in $\mathbb{N} \cup P_{\mathcal{I}}$ and $p \in U$. Let $q \in U$. If $q \in A(\frac{\varepsilon}{2})$ then $|x_q - f_x(p)| < \frac{\varepsilon}{2} < \varepsilon$ which, by definition of f_x, gives us that $|f_x(q) - f_x(p)| < \varepsilon$. If $q \in A(\frac{\varepsilon}{2})^*$ then $\{n \in \mathbb{N} \colon |x_n - f_x(q)| < \frac{\varepsilon}{2}\} \in q$, and $\{n \in \mathbb{N} \colon |x_n - f_x(p)| < \frac{\varepsilon}{2}\} \in q$. So, there is $n_0 \in \{n \in \mathbb{N} \colon |x_n - f_x(q)| < \frac{\varepsilon}{2}\} \cap \{n \in \mathbb{N} \colon |x_n - f_x(p)| < \frac{\varepsilon}{2}\}$, and finally we have

$$|f_x(p) - f_x(q)| \leq |f_x(p) - x_{n_0}| + |x_{n_0} - f_x(q)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now let us define a function $\Phi\colon l^\infty(\mathcal{I})/c_0(\mathcal{I}) \to C(P_{\mathcal{I}})$ by the formula $\Phi([x]_\sim) = f_{x \restriction P_{\mathcal{I}}}$. Observe that if $y \in [x]_\sim$ then $\Phi([x]_\sim) = \Phi([y]_\sim)$, hence Φ is well defined. Observe also that Φ is a bijection. Indeed, if $f \in C(P_{\mathcal{I}})$ then since $P_{\mathcal{I}}$ is a closed set and $\beta\mathbb{N}$ is a normal space, we can, by the Tietze theorem, extend f to a continuous function $F\colon \beta\mathbb{N} \to \mathbb{R}$. Let $x_n = F(n)$ for $n \in \mathbb{N}$. Then the sequence $x = (x_n)$ is bounded in usual sense, and therefore x is also $\mathcal{I}$-bounded. By the density of $\mathbb{N}$ in $\beta\mathbb{N}$, it is clear that $\Phi([x]_\sim) = f$, so Φ is a surjection. The function Φ is an injection, because by (1) the following equivalence holds:

$$(\forall x, y \in l^\infty(\mathcal{I}))\ (x \sim y \Leftrightarrow f_{x-y}(p) = 0 \text{ for every } p \in P_{\mathcal{I}}).$$

Now, let us show that $||[x]_\sim||^\infty_{\mathcal{I}} = \sup_{p \in P_{\mathcal{I}}} |p^*\text{-}\lim x|$. Denote

$$A := \{\lambda > 0 \colon (\exists K \in \mathcal{I}^*)\ (\forall n \in K)\ |x_n| \leq \lambda\},$$

$$B := \{t \geq 0 \colon (\exists p \in P_{\mathcal{I}}) |p^*\text{-}\lim x| = t\}.$$

It is clear that

$$B := \{t \geq 0\colon (\exists \delta \in \{-1,1\})(\exists p \in P_{\mathcal{I}})\ (\forall \varepsilon > 0)\ \{n \in \mathbb{N}\colon |x_n - \delta t| \leq \varepsilon\} \in p\}.$$

We will prove that $\lambda_0 := \inf A = \sup B =: t_0$. Let $t > \lambda_0$, $\delta \in \{-1,1\}$ and $p \in P_{\mathcal{I}}$ be arbitrary. Then there is a $K \in \mathcal{I}^*$ such that $|x_n| \leq (\lambda_0 + t)/2$ for every $n \in K$. So $L = \{n \in \mathbb{N}\colon |x_n - \delta t| \leq (t - \lambda_0)/3\}$ satisfies $L \subset \mathbb{N} \setminus K \in \mathcal{I}$. Hence $L \notin p$ so $t \notin B$, and $t_0 \leq \lambda_0$ follows.

Assume now that $\lambda > t_0$. Fix $p \in P_{\mathcal{I}}$ and $\varepsilon \in (0, \lambda - t_0)$. There exist $t \in [0, t_0]$ and $\delta \in \{-1,1\}$ such that $\{n \in \mathbb{N}\colon |x_n - \delta t| \leq \varepsilon\} \in p$. Observe that

$$\{n \in \mathbb{N}\colon |x_n - \delta t| \leq \varepsilon\} \subset \{n \in \mathbb{N}\colon |x_n| \leq t + \varepsilon\}$$

$$\subset \{n \in \mathbb{N}\colon |x_n| \leq t_0 + \varepsilon\} \subset K_0 := \{n \in \mathbb{N}\colon |x_n| \leq \lambda\} \in p.$$

By (1) we have that $K_0 \in \mathcal{I}^*$, consequently $\lambda_0 \leq t_0$.

Ending the proof, since Φ is obviously a linear operator, we infer that Φ is an isometrical isomorphism. □

Corollary 2. *Let $\mathcal{I} = \mathcal{I}_f$, thus $P_{\mathcal{I}} = \beta\mathbb{N} \setminus \mathbb{N}$ and $c_0(\mathcal{I}) = c_0$ – the set of all sequences convergent in usual sense to zero. Then the spaces $l^\infty(\mathcal{I})/c_0$ and $C(\beta\mathbb{N} \setminus \mathbb{N})$ are isometrically isomorphic, that is why $l^\infty(\mathcal{I})/c_0$ is a Banach space.*

The above fact can be also deduced from [S, Theorem 4.2.2, p.77].

3. On the equality $W(\mathcal{I}) = c^*(\mathcal{I})$

In this part of the paper, we will investigate problems connected with the notions of $\mathcal{I}$-variation and $\mathcal{I}$-convergence. We give a consistent positive answer to the following question asked in [FGT] (in fact we prove something more):

"Does there exist an admissible ideal $\mathcal{I}$ such that $W(\mathcal{I}) = c^*(\mathcal{I})$?"

From now on, filters on $\mathbb{N}$ will be denoted by $\mathcal{F}$. So, $\mathcal{F}^*$ denotes the dual ideal of a filter $\mathcal{F}$. For any set A, by $[A]^{\mathbb{N}}$ we denote the set of all countably infinite subsets of A. All filters we will consider, contain the Fréchet filter $\mathcal{I}_f^*$ (i.e. the filter consisting of all co-finite subsets of $\mathbb{N}$). Note that an ideal is admissible if and only if its dual filter contains the Fréchet filter. Assume that a set $A \subset \mathbb{N}$ has infinite intersection with every set from $\mathcal{F}$. Then by $\langle \mathcal{F}, A \rangle$ we denote the filter generated by $\mathcal{F} \cup \{A\}$. We say that a filter $\mathcal{F}$ is κ-generated if there is a family $\{A_\alpha : \alpha < \kappa\}$ of subsets of $\mathbb{N}$ such that $\mathcal{I}_f^* \cup \{A_\alpha : \alpha < \kappa\}$ generates $\mathcal{F}$.

We say that X is *almost contained* in Y (and write $X \subset^* Y$) if $X \setminus Y$ is finite. An ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called a *P-ideal* if for every sequence $(A_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subset^* A$ for every n. An ultrafilter $\mathcal{F}$ is called a *P-point* if for any partition (R_n) of $\mathbb{N}$, either there is n with $R_n \in \mathcal{F}$ or there is $U \in \mathcal{F}$ such that $|R_n \cap U| < \omega$ for all n. It is easy to see that the dual ideal $\mathcal{F}^\star$ to a P-point $\mathcal{F}$ is a maximal P-ideal. By [KSW, Theorem 3.2], $\mathcal{I}$ is a P-ideal if and only if $c^*(\mathcal{I}) = c(\mathcal{I})$ (in [KSW], instead of P-ideals, ideals with property (AP) are considered but these two notions coincide). By [FGT, Proposition 3], $\mathcal{I}$ is a maximal ideal if and only if $c(\mathcal{I}) = l^\infty(\mathcal{I})$. Hence $\mathcal{I}$ is a maximal P-ideal if and only if $c^*(\mathcal{I}) = c(\mathcal{I}) = l^\infty(\mathcal{I})$. Note that the existence of P-points is independent of ZFC, see [W].

The pseudo-intersection number is defined as follows [V]:

$$\mathfrak{p} = \min\{|\mathcal{A}| : \mathcal{A} \subset [\mathbb{N}]^{\mathbb{N}},\ \mathcal{A} \text{ has SFIP and } \neg(\exists X \in [\mathbb{N}]^{\mathbb{N}})(\forall Y \in \mathcal{A})\ X \subset^* Y\}.$$

Here SFIP stands for *strong finite intersection property* which means that every finite subset of $\mathcal{A} \subset [\mathbb{N}]^{\mathbb{N}}$ has infinite intersection. Note that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$. Consistently, these inequalities can be strict, or $\mathfrak{p} = \omega_1$ or $\mathfrak{p} = \mathfrak{c}$. In the sequel we will use the fact that $\mathfrak{p} = \mathfrak{c}$ is consistent (for instance, it holds under CH or MA).

Proposition 3. *Assume that a filter $\mathcal{F}$ is κ-generated for some $\kappa < \mathfrak{p}$. If $K \in [\mathbb{N}]^{\mathbb{N}}$ with $K \notin \mathcal{F}^*$, then there is $L \subset K$ such that $[L]^{\mathbb{N}} \cap \mathcal{F}^* = \emptyset$.*

Proof. Let $\{A_\alpha : \alpha < \kappa\}$ be a family of subsets of $\mathbb{N}$ which generates $\mathcal{F}$. Then every set in $\mathcal{F}$ is a superset of $\bigcap_{\alpha\in F} A_\alpha \setminus G$ for some finite sets F, G. Since

$K \notin \mathcal{F}^*$, it follows that $K \cap \bigcap_{\alpha \in F} A_\alpha$ is infinite for every finite F. By $\kappa < \mathfrak{p}$ we can pick an infinite $L \subset K$ almost contained in every $K \cap A_\alpha$, $\alpha < \kappa$. Hence each infinite subset of L has the same property. So $[L]^{\mathbb{N}} \cap \mathcal{F}^* = \emptyset$. □

Theorem 4. *Assume that $\mathfrak{p} = \mathfrak{c}$. Let $\tau < \mathfrak{p}$. Suppose that $\mathcal{B}_1$, $\mathcal{B}_2$ are two properties of sequences $x \in \mathbb{R}^{\mathbb{N}}$ such that:*

(a) *for all $x \in \mathbb{R}^{\mathbb{N}}$ and $K \in [\mathbb{N}]^{\mathbb{N}}$, if $x_{\restriction K}$ has $\mathcal{B}_1$, then there is $L \in [\mathbb{N}]^{\mathbb{N}}$, $L \subset K$, such that $x_{\restriction L}$ has $\mathcal{B}_2$;*

(b) *$\mathcal{B}_1$ is closed under taking subsequences, i.e. for all $x \in \mathbb{R}^{\mathbb{N}}$, $L, K \in [\mathbb{N}]^{\mathbb{N}}$, if $L \subset K$ and $x_{\restriction K}$ has $\mathcal{B}_1$, then $x_{\restriction L}$ has $\mathcal{B}_1$.*

If a filter $\mathcal{F}$ is τ-generated, then $\mathcal{F}$ can be extended to a filter $\mathcal{F}'$ such that for any $x \in \mathbb{R}^{\mathbb{N}}$ and $K \in \mathcal{F}'$, if $x_{\restriction K}$ has $\mathcal{B}_1$, then there is $L \in \mathcal{F}'$, $L \subset K$, such that $x_{\restriction L}$ has $\mathcal{B}_2$.

Proof. Define $\mathcal{K} = \{(x, K) \in \mathbb{R}^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}} : x_{\restriction K} \text{ has } \mathcal{B}_1\}$. If $\mathcal{K}$ is empty, then the assertion trivialy holds. Otherwise by (b) the family $\mathcal{K}$ has cardinality $\mathfrak{c}$. Let $\mathcal{K} = \{(x_\alpha, K_\alpha) : \alpha < \mathfrak{c}\}$. We will define a family $\{\mathcal{F}_\alpha : \alpha < \mathfrak{c}\}$ of filters on $\mathbb{N}$ such that:

(i) $\mathcal{F}_0 = \mathcal{F}$;

(ii) $\mathcal{F}_\alpha$ is $\tau + |\alpha|$-generated;

(iii) $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ for $\alpha < \beta$;

(iv) if $K_\alpha \notin (\bigcup_{\gamma<\alpha} \mathcal{F}_\gamma)^*$, then there is $L \subset K_\alpha$ such that $L \in \mathcal{F}_\alpha$ and $(x_\alpha)_{\restriction L}$ has $\mathcal{B}_2$.

Step α. If $K_\alpha \in (\bigcup_{\gamma<\alpha} \mathcal{F}_\gamma)^*$, then by Proposition 3 there is an infinite set $L \subset \mathbb{N} \setminus K_\alpha$ with $[L]^{\mathbb{N}} \cap (\bigcup_{\gamma<\alpha} \mathcal{F}_\gamma)^* = \emptyset$. Put $\mathcal{F}_\alpha = \langle \bigcup_{\gamma<\alpha} \mathcal{F}_\gamma, L \rangle$. If $K_\alpha \notin (\bigcup_{\gamma<\alpha} \mathcal{F}_\gamma)^*$, then by Proposition 3 there is $L' \subset K_\alpha$ with $[L']^{\mathbb{N}} \cap (\bigcup_{\gamma<\alpha} \mathcal{F}_\gamma)^* = \emptyset$. By (b) the sequence $(x_\alpha)_{\restriction L'}$ has $\mathcal{B}_1$. Hence by (a) there is infinite $L \subset L'$ such that $(x_\alpha)_{\restriction L}$ has $\mathcal{B}_2$. Put $\mathcal{F}_\alpha = \langle \bigcup_{\gamma<\alpha} \mathcal{F}_\gamma, L \rangle$.

Let $\mathcal{F}' = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$. By the construction, for each $x \in \mathbb{R}^{\mathbb{N}}$ with $x_{\restriction K}$ having $\mathcal{B}_1$ for some $K \in \mathcal{F}'$, there exists $L \in \mathcal{F}'$ such that $x_{\restriction L}$ has $\mathcal{B}_2$. □

The following corollary gives a consistent positive answer to the problem posed by Faisant, Grekos and Toma in [FGT, Remark 7].

Corollary 5. *It is independent of ZFC that there is an ideal $\mathcal{I}$ with $l^\infty(\mathcal{I}) = M(\mathcal{I})$. In particular it is consistent that $W(\mathcal{I}) = c^*(\mathcal{I})$ for some $\mathcal{I}$.*

Proof. Let $\mathcal{B}_1$ denote the property of being bounded and let $\mathcal{B}_2$ denote the property of being monotone. Starting with the Fréchet filter $\mathcal{I}_f^*$ and assuming that $\mathfrak{p} = \mathfrak{c}$, by Theorem 4 we find $\mathcal{F}' \supset \mathcal{I}_f^*$ with $l^\infty(\mathcal{F}'^*) \subset M(\mathcal{F}'^*)$.

As we have mentioned above, $l^\infty(\mathcal{I}) = M(\mathcal{I})$ implies that $\mathcal{I}^*$ is a P-point. Hence in Shelah's model in which there is no P-points (see [W]), equality $l^\infty(\mathcal{I}) = M(\mathcal{I})$ cannot hold for any ideal $\mathcal{I}$. □

One can ask if the problem raised by Faisant, Grekos and Toma is decidable in ZFC, and how many monotone subsequences there are in a bounded sequence.

Problem 6. *Does there exist in ZFC an ideal $\mathcal{I}$ with $W(\mathcal{I}) = c^*(\mathcal{I})$?*

Problem 7. *Assume that $W(\mathcal{I}) = c^*(\mathcal{I})$. Is it true that $M(\mathcal{I}) = c^*(\mathcal{I})$?*

Another problem posed in [FGT, Remark 4] is the following. Assume that $W(\mathcal{F}^*) \subsetneq c^*(\mathcal{F}^*)$. Does there exist $F \notin \mathcal{F}^*$ such that

$$W(\langle \mathcal{F}, F\rangle^*) \subsetneq c^*(\langle \mathcal{F}, F\rangle^*)?$$

We give a partial answer.

Proposition 8. *Let $\kappa < \mathfrak{p}$. Assume that a filter $\mathcal{F}$ is κ-generated. Then $W(\mathcal{F}^*) \subsetneq c^*(\mathcal{F}^*)$.*

Proof. Let $\{A_\alpha : \alpha < \kappa\}$ be a set of generators of $\mathcal{F}$. Find $L \in [\mathbb{N}]^{\mathbb{N}}$ such that $L \subset^* A_\alpha$ for any $\alpha < \kappa$. Let $L = \{l_1 < l_2 < \ldots\}$. Define $x_n = 0$ if $n \notin L$, and $x_n = (-1)^i/i$ if $n = l_i \in L$. Since x converges to 0, then $x \in c^*(\mathcal{F}^*)$. On the other hand, if $K \in \mathcal{F}$, then $L \subset^* K$. Hence $\operatorname{Var} x_{\restriction K} = \infty$ and $x \notin W(\mathcal{F}^*)$. □

Proposition 8 gives a sufficient condition for a strict inclusion $W(\mathcal{I}) \subsetneq c^*(\mathcal{I})$. This is not a necessary condition. Indeed, let $\mathcal{I}_d$ stand for the ideal subsets of $\mathbb{N}$ of asymptotic density zero, see [FGT]. Then the sequence $((-1)^n/n)$ is a witness for $W(\mathcal{I}_d) \subsetneq c^*(\mathcal{I}_d)$ (see [FGT, Proposition 6]). By [LV, Theorem 1], $\mathcal{I}_d$ is not τ-generated for any $\tau < \mathfrak{d}$. Since $\mathfrak{p} \leq \mathfrak{d}$, then $\mathcal{I}_d$ is not τ-generated for any $\tau < \mathfrak{p}$ (we refer the reader to [V] for definition of $\mathfrak{d}$ and its properties).

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